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## Abstract

Evaluating uncertainty of traffic networks is very important. In order to assess the uncertainty theoretically, we need an equilibrium model that can estimate probabilistic distributions of (link) travel times or traffic volumes. Stochastic user equilibrium (SUE) can calculate deterministic travel times, but cannot calculate variance or volatility of travel times. SUE seemed to be insufficient for assessing network's uncertainty. We develop a stochastic network equilibrium model, in which travel times and traffic volumes are random variables. The equilibrium model can estimate variances of link travel times, and evaluate the network's uncertainty. Furthermore, the model can be extended for uncertain demands by introducing hypothetical links. Thus, this equilibrium model opens the door for modelling traffic networks in uncertain environments, especially uncertain demands.

## Keywords

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# A TRAFFIC NETWORK EQUILIBRIUM MODEL FOR UNCERTAIN DEMANDS

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## ABSTRACT

Evaluating uncertainty of traffic networks is very important. In order to assess the uncertainty theoretically, we need an equilibrium model that can estimate probabilistic distributions of (link) travel times or traffic volumes. Stochastic user equilibrium (SUE) can calculate deterministic travel times, but cannot calculate variance or volatility of travel times. SUE seemed to be insufficient for assessing network's uncertainty. We develop a stochastic network equilibrium model, in which travel times and traffic volumes are random variables. The equilibrium model can estimate variances of link travel times, and evaluate the network's uncertainty. Furthermore, the model can be extended for uncertain demands by introducing hypothetical links. Thus, this equilibrium model opens the door for modeling traffic networks in uncertain environments, especially uncertain demands.

## 1. INTRODUCTION

Stochastic User Equilibrium (SUE) introduced by Daganzo and Sheffi (*1*) is one of the most important network equilibrium. SUE is regarded as Wardrop's equilibrium (*2*) with route choice based on random utility models. The (route) utility in route choice of SUE has an error term. The interpretation of the error term is disputable. Variance of the error term is constant despite of route's length, and the term does not seem to reflect variation or uncertainty of travel time on the route. The error term should be interpreted as "perceptual" error or effect of the components that are not considered in the model. Furthermore, network flows in SUE is not stochastic but deterministic. SUE cannot treat uncertainty or variation of network flows.

We assume that a driver chooses a route stochastically, but does not chooses the route based on random utility theory; just chooses the route "probabilistically." This represents a combination of choices with probabilities. For example, Choice 1 is adopted with probability 0.5 and Choice 2 with 0.5. This type of choice is called as mixed strategy in game theory. Under mixed-strategy-type stochastic route choice, the route flow follows binomial distribution (or multinomial distribution). In this study, it is assumed that route choice is not only stochastic, but route flows are also stochastic. This is fundamentally different from SUE. Furthermore, we introduce hypothetical links (*3*, *4*) which represents no travel to the network. This enables us to consider uncertainty of travel demands because flow of the hypothetical link is also stochastic. The aim of the study is to formulate a stochastic network equilibrium model with stochastic flows under uncertain demands.

## 2. SIMPLE LITERATURE REVIEW

Uncertainty or variation of networks is very important on transportation planning or transportation engineering. There have been several studies about uncertainty of network flows. Mirchandani & Soroush (*5*) assumed that free-flow travel time is random, and proposed a network equilibrium model with probabilistic travel times. Chen et al. (*6*) examined capacity reliability, considering random capacities and demands. Arnott et al. (*7*) also introduced random capacity to network equilibrium. These three studies assumed exogenous randomness. Cascetta (*8*) and Cascetta & Canterella (*9*) formulated day-to-day dynamics of network flows as Markov process. The convergent distribution of network flow could be interpreted as network equilibrium with stochastic flow. The problems of Markov models are how a transition probabilistic matrix (of Markov process) is made and applicability to a large network. Watling (*10*) extended SUE and presented a second order stochastic network equilibrium. He assumed route choice based on random utility theory and stochastic flow variables. The travel demands are assigned based on the mean cost. In this study, we do not assume route choice based on random utility theory, and formulate a stochastic network model more simply. Then, we incorporate driver's risk attitude and uncertain demands into the model. This model enable us to examine network reliability under uncertain demands.

## 2. FOUNDAMENTAL CONCEPT AND FORMULATION

The concept of Wardrop's equilibrium is as follows:

> *Equilibrium under condition that no driver can reduce his route cost by unilaterally switching routes*

This Wardrop's equilibrium is a kind of Nash equilibrium (*11*) in game theory (*12*). Assuming that the driver chooses the route stochastically and that network flows are also stochastic, the following equilibrium condition could be defined as:

> *No driver can reduce his EXPECTED route cost (or utility) by unilaterally switching routes*

This concept is very natural in light of generalization of Wardrop's equilibrium. Nakayama (*13*) and Bell & Cassir (*14*) also touched upon the above equilibrium condition. Stochastic choice in this study is a combination of choices with probabilities. For example, Choice 1 is adopted with probability 0.5 and Choice 2 with 0.5. As mentioned before, this is called mixed strategy in game theory.

Let $i$ ($i$ = 1, 2,…, I) denote an origin-destination (OD) pair of the network and $N^i$ the demand of OD pair $i$. Let $j$ denote a route and the number of routes is $J$, and the set of routes between OD pair $i$ is $J^i$. Let $a$ denote a link and the number of links is $A$. Assume that the driver who travels between OD pair $i$ chooses Route $j$ with probability $p_j$, where $p_j$ is the probability of choosing Route $j$. Especially, if Route $j$ belongs to OD pair $i$, the probability of choosing Route $j$ is denoted by $p^i_j$. Then, the route flows of OD pair $i$ follow a multinomial distribution. The flow on Route $j$ of OD pair $i$ follows a binomial distribution, Bin($N^i$, $p^i_j$). Thus, the flow is a random variable and the travel time is also a random variable.

The equilibrium model of this study is formulated as follows:

$$\begin{cases} \mathrm{E}[U^i_j] = \lambda^i & if\ p^i_j > 0 \\ \mathrm{E}[U^i_j] \geq \lambda^i & if\ p^i_j = 0 \end{cases} \qquad \forall j\ \forall i \tag{1}$$

where E[·] denotes the operator of expectation, $U_j^i$ the random variable of utility on Route $j$ connecting OD pair $i$, and $\lambda^i$ the minimum expected utility between OD pair $i$. In this study, $U_j^i$ is defined as E[$T_j^i$] + $\eta$·Var[$T_j^i$], where $\eta$ is a risk attitude parameter, Var[·] denotes the operator of taking variance, and $T_j^i$. is the random variable of travel time on Route $j$. The driver is risk-averse when the parameter, $\eta$, is positive and he is risk-prone when it is negative.

The above equations can be formulated as a non-linear complementary problem:

$$\text{Determine } \mathbf{x}^* = (\mathbf{p}^*, \lambda^*) \in R^J_+ \times R^I_+ \quad \text{such that } \mathbf{x} \cdot \mathbf{F}[\mathbf{x}] = 0, \quad \mathbf{x} \geq \mathbf{0}, \mathbf{F}[\mathbf{x}] \geq \mathbf{0} \tag{2}$$

where $\mathbf{x} \cdot \mathbf{y}$ denotes the inner product of $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x} = (\mathbf{p}, \lambda)$, $\mathbf{F}[\mathbf{x}] = (\mathbf{E}[\mathbf{U}] - \mathbf{\Gamma}^{\mathrm{T}} \lambda, \mathbf{\Gamma}\,\mathbf{p} - \mathbf{I})$, $\mathbf{E}[\mathbf{U}]$ is the vector of the expected utilities (which is the function of $\mathbf{p}$), $\mathbf{p}$ the vector of the route choice probabilities, $\lambda$ the vector of the minimum expected utilities, $\mathbf{\Gamma}$ the J×I route-OD incident matrix, $\mathbf{I}$ a unit vector, and $\mathbf{0}$ a null vector.

Alternatively, we can formulate the equations as a variational inequality problem:

$$\text{Determine } \mathbf{x}^* = (\mathbf{p}^*, \lambda^*) \in R^J_+ \times R^I_+ \quad \text{such that } \mathbf{F}[\mathbf{x}^*] \cdot (\mathbf{x} - \mathbf{x}^*) \geq 0 \quad \forall \mathbf{x} \tag{3}$$

# 3. EXPECTED TRAVEL TIME

Traffic volume on each route follows a binomial distribution, and the link volume which consists of the demand between one OD pair also follows a binomial distribution. In general, OD pairs are multiple, not single, and the link volume is the summation of binomial variables. The summation of binominal variables does not become a binominal variable like normal distributions, and treating binomial distributions is more or less difficult. The expected travel time on the link is calculated as follows:

$$E[T_a] = \sum_{x_a^1=0}^{N^1} \cdots \sum_{x_a^i=0}^{N^i} \cdots \sum_{x_a^I=0}^{N^I} t_a\left(\sum_{i=1}^{I} x_a^i\right) \cdot \prod_{i=1}^{I} \Pr[x_a^i] \tag{4}$$

where $T_a$ is the random variable of travel time on Link $a$, $\Pr[x^i_a]$ is the probability that the link volume of the demand between OD pair $i$ is $x^i_a$ and is calculated by ${}_{N^i}C_{x_a^i}\,(p_a^i)^{x_a^i} \cdot (1-p_a^i)^{N^i - x_a^i}$, $p^i_a$ is $\sum_{j \in J^i} \delta_{a,j} \cdot p_j^i$ and represents the probability that the driver between OD pair $i$ travels on Link $a$, and $\delta_{a,j}$ is 1 if Link $a$ is part of Route $j$; otherwise, 0.

In this study, we adopt a BPR-type cost-flow performance function for calculating travel time, and link travel time can be expressed as $\alpha + \beta \cdot x^\gamma$, where $x$ is link volume and $\alpha$, $\beta$, and $\gamma$ are positive constant parameters. When $\gamma$ is an integer (usually 4.0 is used), the expected link travel time can be calculated using moment generating functions. A moment generating function, $M(s)$, is defined as $\mathrm{E}[e^{sX}]$ (e.g., *15*, *16*). As a property of the moment generating function, $\mathrm{E}[X^\gamma] = \left.\frac{d^\gamma M(s)}{ds^\gamma}\right|_{s=0}$. Also, the moment generating function of the summation of independent random variables is the product of the moment generating function of each random variable.

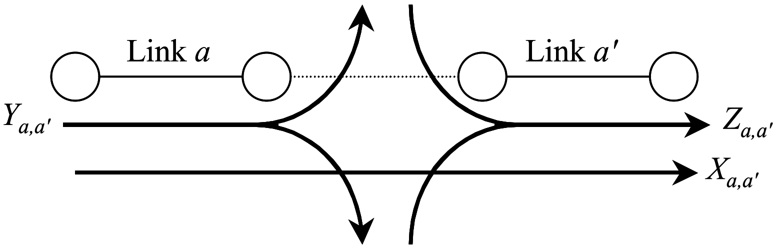



**Fig. 1 Decomposition of link volumes for covariance**

The random variable of link volume, $X_a$, is $\sum_j \delta_{a,j} \cdot F_j$, where $F_j$ is a binomial variable of the flow on Route $j$. Let $M^i_a(s)$ denote a moment generating function of link flow between OD pair $i$. Then, the moment generating function of link flow of all demands, $M_a(s)$, is $\prod_i M^i_a(s)$. The expected link travel time is:

$$\mathrm{E}[T_a] = \alpha + \beta \cdot \left. \frac{d^\gamma M_a(s)}{ds^\gamma} \right|_{s=0} \tag{5}$$

The expected travel time of Route $j$, $\mathrm{E}[T_j]$, is $\sum_a \delta_{a,j} \cdot \mathrm{E}[T_a]$. Variance of link travel time, $\mathrm{Var}[T_a]$, is $\mathrm{E}[T_a^2] - \{\mathrm{E}[T_a]\}^2$.

Covariance of travel times of Link $a$ and Link $a'$, $\mathrm{Cov}[T_a, T_{a'}]$, is also calculated using moment generating functions. In order to calculate covariance of travel times between links, traffic volumes on Link $a$ and Link $a'$ are decomposed as illustrated in Figure 1. Let $X_{a,a'}$ denote the random variable of traffic volume that flows on both Link $a$ and Link $a'$, $Y_{a,a'}$ the variable of the volume that flows on Link $a$ only, but does not flow on Link $a'$, and $Z_{a,a'}$ the variable of the volume that flows on Link $a'$. Covariance of travel times between links is $\mathrm{E}[t_a(X_{a,a'} + Y_{a,a'}) \cdot t_{a'}(X_{a,a'} + Z_{a,a'})] - \mathrm{E}[T_a] \cdot \mathrm{E}[T_{a'}]$. The covariance can be calculated using the following equation as follows:

$$\mathrm{Cov}[T_a, T_{a'}] = \sum_{x^1_{a,a'}=0}^{N^1} \cdots \sum_{x^I_{a,a'}=0}^{N^I} \sum_{y^1_{a,a'}=0}^{N^1} \cdots \sum_{y^I_{a,a'}=0}^{N^I} \sum_{z^1_{a,a'}=0}^{N^1} \cdots \sum_{z^I_{a,a'}=0}^{N^I} t_a\left( \sum_{i=1}^{I} x^i_{a,a'} + \sum_{i=1}^{I} y^i_{a,a'} \right) \cdot t_{a'}\left( \sum_{i=1}^{I} x^i_{a,a'} + \sum_{i=1}^{I} z^i_{a,a'} \right) \cdot \prod_{i=1}^{I} \Pr[x^i_{a,a'}, y^i_{a,a'}, z^i_{a,a'}] \tag{6}$$

where $x^i_{a,a'}$, $y^i_{a,a'}$, and $z^i_{a,a'}$ are the values of $X^i_{a,a'}$, $Y^i_{a,a'}$, and $Z^i_{a,a'}$, respectively, $X^i_{a,a'}$, $Y^i_{a,a'}$, and $Z^i_{a,a'}$ are the values of $X^i_{a,a'}$, $Y^i_{a,a'}$, and $Z^i_{a,a'}$ (which consist of the demand between OD pair $i$), $\Pr[x^i_{a,a'}, y^i_{a,a'}, z^i_{a,a'}]$

is $\frac{N^i!}{x^i_{a,a'}!y^i_{a,a'}!z^i_{a,a'}!}(p^i_{x_{a,a'}})^{x^i_{a,a'}} \cdot (p^i_{y_{a,a'}})^{y^i_{a,a'}} \cdot (p^i_{z_{a,a'}})^{z^i_{a,a'}} \cdot (1-p^i_{x_{a,a'}} - p^i_{y_{a,a'}} - p^i_{z_{a,a'}})^{N^i - x^i_a}$, and $p^i_{x_{a,a'}}$ denotes the probability that a driver of the demand between OD pair *i* takes the routes including both Link *a* and Link *a'*, that is, the probability of $x_{a,a'}$, $p^i_{y_{a,a'}}$ the probability of $y_{a,a'}$, $p^i_{z_{a,a'}}$ the probability of $z_{a,a'}$.

Cov[$T_a$, $T_{a'}$] can also be calculated using the moment generating function. At first, let us calculate covariance of link volumes which consist of the demand between OD pair *i*. In this case, $X^i_{a,a'}$, $Y^i_{a,a'}$, and $Z^i_{a,a'}$ follow a multinomial distribution (quadrinomial distribution). Let $M^i(s, t, u)$ denote a moment generating function of this quadrinomial distribution. $M^i(s, t, u)$ is $(p^i_x \cdot e^s + p^i_y \cdot e^t + p^i_z \cdot e^u + 1 - p^i_x - p^i_y - p^i_z)^N$ where the subscript, *a,a'*, is omitted. When there are multiple OD pairs, the moment generating function of all demands, $M(s, t, u)$, is $\prod_i M^i(s,t,u)$. As described above, the covariance is E[$t_a(X_{a,a'} + Y_{a,a'}) \cdot t_{a'}(X_{a,a'} + Z_{a,a'})$] – E[$T_a$] · E[$T_{a'}$], and is written as a polynomial equation of E[$X^l$, $Y^m$, $Z^n$] ($0 \le l, m, n \le 2\gamma$). E[$X^l$, $Y^m$, $Z^n$] is calculated by $\left.\frac{\partial^{l+m+n} M(s,t,u)}{\partial s^l \partial t^m \partial u^n}\right|_{s,t,u=0}$ (*16*). Therefore, the covariance can be calculated more efficiently using the moment generating function than the equation (6).

Variance of the route travel time, Var[$T_j$], is calculated as follows:

$$\mathrm{Var}[T_j] = \sum_a \delta_{a,j} \cdot \mathrm{Var}[T_a] + \sum_a \sum_{a'} \sum_{a \neq a'} \delta_{a,j} \cdot \delta_{a',j} \cdot \mathrm{Cov}[T_a, T_{a'}] \tag{7}$$

# 4. UNCERTAINTY OF TRAVEL DEMANDS

The equilibrium model proposed above assesses uncertainty of network flow, which occurs because of stochastic route choice. One of the main causes of uncertainty in transportation networks is uncertain travel demands. We can easily incorporate uncertain travel demands into our equilibrium model.

Let us introduce a latent travel demand, $\tilde{N}$. The latent travel demand represents the number of drivers who have possibility of making trips, and is defined for each OD pair. Let $p^i_0$ denote the probability that the latent travel demand do not actually make trips. The actual travel demand of OD pair *i*, which represents the number of drivers that make trips actually, follows the binomial distribution, Bin($\tilde{N}^i$, $1 - p^i_0$). Thus, uncertain travel demand of each OD pair can be expressed as the binomial distribution. As noted earlier, our equilibrium model is based on binomial distributions, and we can easily incorporated uncertain travel demands into the model, adding hypothetical link which represents the choice of not making a trip.

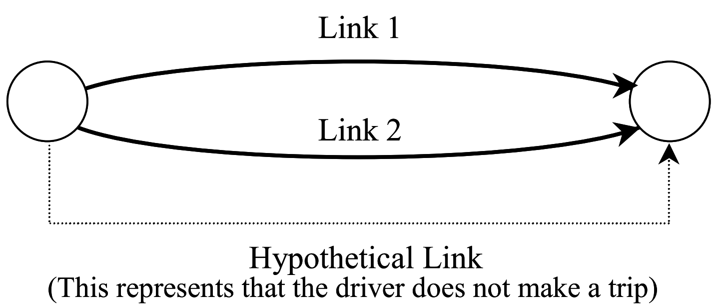



**Figure 2 The example network**

## 5. EXAMPLE

As an illustration of the equilibrium model, a simple network example is presented, consisting of a single OD pair and two parallel links (routes). Figure 2 shows the network. As the value of $\gamma$ in the performance function, $\alpha + \beta \cdot x^{\gamma}$, 2.0 is used on both links. The values of $\alpha$ and $\beta$ of Link 1 are 10 and $1/1000^2$, respectively. These represent that the capacity and free-flow link travel time are 1000 (vehicles per hour) and 10 (minutes). The values of Link 2 are 20 and $1/2000^2$, and represent that Link 2 has the capacity of 2000 and the free-flow travel time of 20. Route utility, $U_j$, is defined as $E[T_j] + \eta \cdot Var[T_j]$. In this example, as the values of the risk attitude parameter, $\eta$, 0.0, 1.0, and 2.0 are used. Table 1 presents the cases in the example.

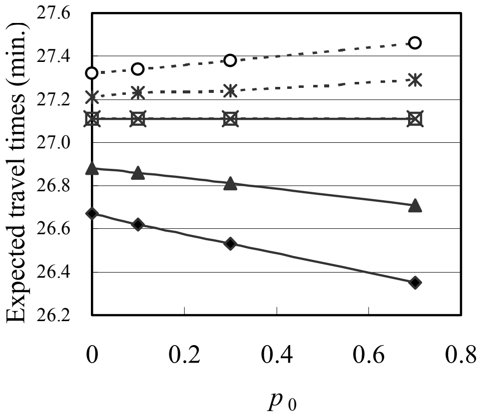
**Table 1 The cases in the example**

| | | Case 1a | Case 1b | Case 1c | Case 1d | Case 2a | Case 2b | Case 2c | Case 2d |
|---|---|---|---|---|---|---|---|---|---|
| Latent demand | $p_0$ | 0 | 0.1 | 0.3 | 0.7 | 0 | 0.1 | 0.3 | 0.7 |
| | $\tilde{N}$ | 2500 | 2778 | 3571 | 8333 | 3500 | 3889 | 5000 | 11667 |
| Actual demand | Average | 2500 | 2500 | 2500 | 2500 | 3500 | 3500 | 3500 | 3500 |
| | Variance | 0 | 277 | 750 | 1750 | 0 | 388 | 1050 | 2450 |

Figure 3 illustrates the expected link travel times with each value of the risk parameter in Case 1a through 1d. Figure 4 presents the expected travel times in Case 2a through 2d. When $\eta = 0.0$ (Case 1a and Case 2a), the expected travel times on both link are equivalent, and in the other cases, the expected travel times on Link 1 is longer than Link 2. This is because Link 1 has a smaller capacity and tends to be more congested than Link 2. The figures show that the difference of the expected travel times on both links is larger as variance of actual travel demand, that is, $p_0$, is larger.

## 6. CONCLUSIONS

Evaluating uncertainty of traffic networks is very important. In order to assess it theoretically, we need



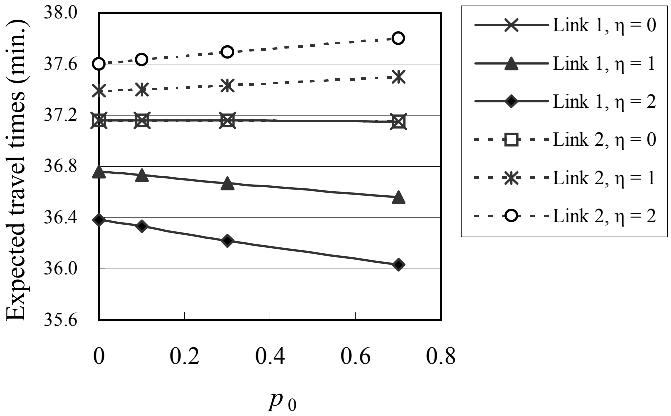




**Figure 3 Expected travel times in Case 1** **Figure 4 Expected travel times in Case 2**

an equilibrium model that can estimate the probabilistic distributions of link travel times. However, stochastic user equilibrium (SUE) can calculate deterministic travel times, but cannot calculate variance or volatility of travel times. SUE is insufficient for assessing network's uncertainty. We proposed the stochastic network equilibrium model, in which travel times and traffic volumes are random variables, and formulated it as a complimentary problem. The model can estimate variances of link travel times, and evaluate uncertainty in the network, considering drivers' risk attitude. Then, we extended the equilibrium model to consider uncertain travel demands using hypothetical links. As for the future directions of the study, we will develop an algorithm for large networks and examine uniqueness of the solution.

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