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Part 2:

Methodologies of Network Flow Analysis

東京大学大学院工学系研究科社会基盤学専攻
交通・都市・国土学研究室

Core references

1. Sheffi, Y.: Urban Transportation Networks, Prentice-Hall, 1985
2. Bell, M. G. H. and Iida, Y.: Transportation Network Analysis, John Wiley & Sons, 1997.
3. Oppenheim, N. Urban Travel Demand Modeling, John Wiley & Sons, 1995.

4. **土木学会編：交通ネットワークの均衡分析，丸善**, 1998.
5. **土木学会編：道路交通需要予測の理論と適用第Ⅰ編，丸善**, 2003.
6. **土木学会編：道路交通需要予測の理論と適用第Ⅱ編，丸善**, 2006.

Approaches in traffic flow analysis

- Euler (Macroscopic) vs Lagrange (Microscopic)
- Static (静的) vs Dynamic (動的)
- Normative (規範的) vs Descriptive (記述的)

Tools for traffic flow analysis

- Equilibrium flow analysis
  - Euler & Normative
  - Mostly Static, rarely dynamic
  - to focus on the “objective state” in NW flow derived by mathematical optimization
- Traffic simulation analysis
  - Euler/Lagrange & Descriptive
  - Dynamic
  - to focus on the “consequent state” in NW flow

What is "equilibrium" (均衡) ?

- Any system in the world finally becomes stable in nature
- Equilibrium is a state of balance, especially between opposing forces or influences
- Equilibrium is a state in which system don't have any motivation to change it (most stable state)

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Behavioral criterion (行動規範) in a transport NW

- Most travelers prefer…
- Fastest route
- Cheapest route
- Route with maximum utility
- Utility maximization (効用最大化)
- Generalized cost minimization (一般化費用最小化)

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What is equilibrium in a transport NW

- Wardrop's rule (1952) (**配分原則**)
  1. For each OD pair, at user equilibrium (**UE**), the travel time on all used paths is equal and less than or equal to the travel time that would be experienced by a single vehicle on any unused path
  2. At system optimum (**SO**), the total travel time of all vehicles is smallest

- Balance between demand (**需要**) and supply (**供給**)

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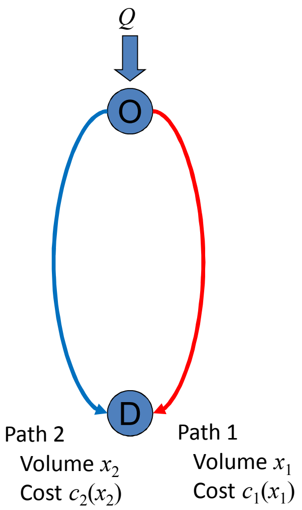
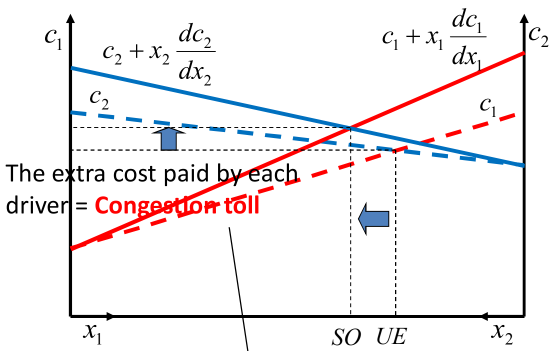
Difference between UE & SO





Market failure (市場の失敗)

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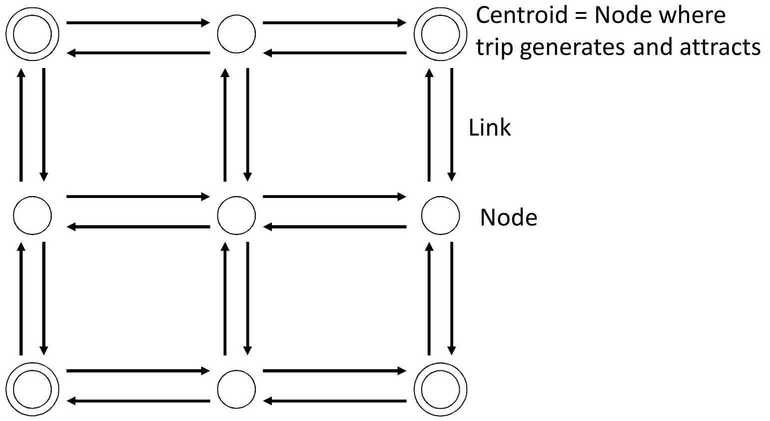
Expression of NW system based on Graph Theory



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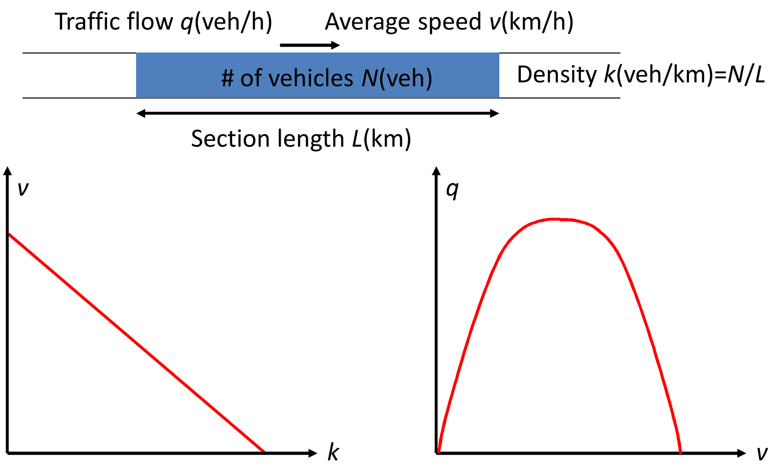
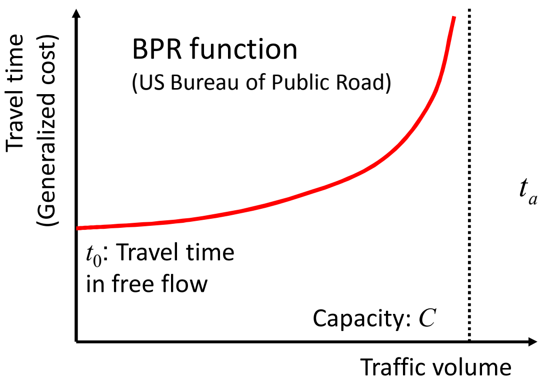
Dependency of travel cost to demand

- Flow dependent
  - Travel cost of a link is affected by demand level (=congestion)
- Flow independent
  - Travel cost of a link is never affected by demand level

Characteristics of link performance in road NW



Link performance function



Parameter

$$t_a(x_a) = t_{0a}\left\{1+\alpha\left(\frac{x_a}{C_a}\right)^{\beta}\right\}$$

Link a

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Formulation of UE(利用者最適)

- For each OD pair, at user equilibrium (**UE**), the travel time (or generalized cost) on all used paths is equal and less than or equal to the travel time that would be experienced by a single vehicle on any unused path

$$\begin{cases} c_k^{rs} = c_{rs} & (f_k^{rs} > 0) \\ c_k^{rs} \geq c_{rs} & (f_k^{rs} = 0) \end{cases} \quad , {}^{\forall}k \in K_{rs} \ , {}^{\forall}rs \in \Omega$$

$$\begin{aligned} s.t. \quad & \sum_{k \in K_{rs}} f_k^{rs} - Q_{rs} = 0 \\ & f_k^{rs} \geq 0 \\ & c_k^{rs} = \sum_{a \in A} \delta_{a,k}^{rs} t_a(x_a) \\ & x_a = \sum_{k \in K_{rs}, rs \in \Omega} \delta_{a,k}^{rs} f_k^{rs} \end{aligned}$$

- f_k^{rs} Traffic flow of path k of OD rs
- c_k^{rs} Travel time of path k of OD rs
- c_{rs} Minimum travel time of OD rs
- Q_{rs} Traffic volume of OD rs
- $\delta_{a,k}^{rs}$ 1 if link a is included in path k of OD rs, otherwise 0

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Conversion to optimization problem:
Beckmann (1956)

$$\min Z_P = \sum_{a \in A} \int_0^{x_a} t_a(w) dw$$

$$\begin{aligned} s.t. \quad & \sum_{k \in K_{rs}} f_k^{rs} - Q_{rs} = 0 \quad , {}^{\forall}k \in K_{rs} \ , {}^{\forall}rs \in \Omega \\ & f_k^{rs} \geq 0 \\ & c_k^{rs} = \sum_{a \in A} \delta_{a,k}^{rs} t_a(x_a) \\ & x_a = \sum_{k \in K_{rs}, rs \in \Omega} \delta_{a,k}^{rs} f_k^{rs} \end{aligned}$$

Constrained Non-linear Optimization Problem

Lagrangian function

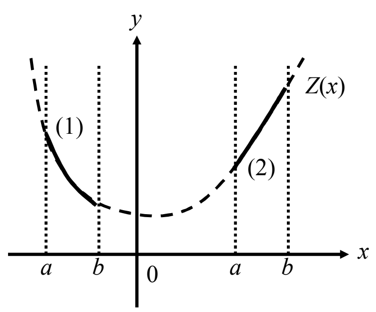
$$L(\mathbf{f},\boldsymbol{\lambda}) = Z_p(\mathbf{f}) - \sum_{rs\in\Omega} \lambda_{rs}\left(\sum_{k\in K_{rs}} f_k^{rs} - Q_{rs}\right)$$

↑ Vector of f_k^{rs} ↑ Lagrange's multiplier

Kuhn-Tucker (optimal) condition

$$\begin{cases} (1)\ f_k^{rs}\dfrac{\partial L(\mathbf{f}^*,\boldsymbol{\lambda}^*)}{\partial f_k^{rs}} = 0 \ \text{ and } \ \dfrac{\partial L(\mathbf{f}^*,\boldsymbol{\lambda}^*)}{\partial f_k^{rs}} \geq 0 \ ,{}^{\forall}k\in K_{rs}\ ,{}^{\forall}rs\in\Omega \\ (2)\ \dfrac{\partial L(\mathbf{f}^*,\boldsymbol{\lambda}^*)}{\partial \lambda_{rs}} = 0 \ ,{}^{\forall}rs\in\Omega \end{cases}$$

Kuhn-Tucker Condition (1)



$$\min Z(x) \quad \text{s.t.} \quad \begin{cases} g_1(x) = x \geq a = d_1 \\ g_2(x) = -x \geq -b = d_2 \end{cases}$$

Primary optimum condition

(1) $x^* = b$

$$\frac{dg_2(x^*)}{dx} = -1 < 0 \quad \text{and} \quad \frac{dZ(x^*)}{dx} = p < 0$$

(2) $x^* = a$

$$\frac{dg_1(x^*)}{dx} = 1 > 0 \quad \text{and} \quad \frac{dZ(x^*)}{dx} = q > 0$$

$$\frac{dZ(x^*)}{dx} = h_n \frac{dg_n(x^*)}{dx}$$

h_n: non-negative scalar

$$\frac{dZ(x^*)}{dx} = \sum_n h_n \frac{dg_n(x^*)}{dx}$$

$$h_n \geq 0 \quad \text{and} \quad h_n\{d_n - g_n(x^*)\} = 0$$

$$g_n(x^*) \geq d_n$$

Secondary optimum condition: Convex set

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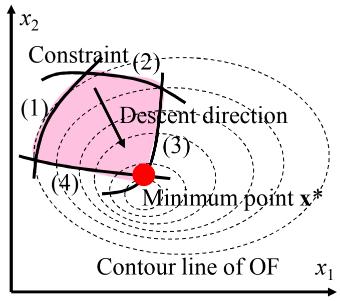
Kuhn-Tucker Condition (2)



$$\min Z(x_1,x_2) \quad \text{s.t.} \quad g_n(x_1,x_2) \ge d_n \quad (n=1,2,3,4)$$

Primary optimum condition

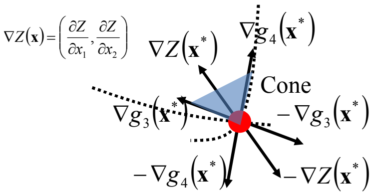
$$\nabla Z(\mathbf{x}^*) = h_3 \nabla g_3(\mathbf{x}^*) + h_4 \nabla g_4(\mathbf{x}^*) \qquad h_3 \ge 0, h_4 \ge 0$$

$$\frac{\partial Z(\mathbf{x}^*)}{\partial x_m} = h_3 \frac{\partial g_3(\mathbf{x}^*)}{\partial x_m} + h_4 \frac{\partial g_4(\mathbf{x}^*)}{\partial x_m} \qquad (m=1,2)$$

$$\frac{\partial Z(\mathbf{x}^*)}{\partial x_m} = \sum_n h_n \frac{\partial g_n(\mathbf{x}^*)}{\partial x_m} \qquad \forall m, \forall n$$

$$h_n \ge 0 \quad \text{and} \quad h_n\{d_n - g_n(x^*)\} = 0$$

$$g_n(x^*) \ge d_n$$



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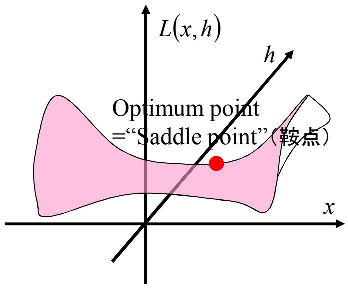
Kuhn-Tucker Condition (3)

$$\min Z(\mathbf{x}) \quad \text{s.t.} \quad g_n(\mathbf{x}) \ge d_n \quad \forall n$$

Lagrange's multiplier

Lagrangian function $\quad L(\mathbf{x},\mathbf{h}) = Z(\mathbf{x}) + \sum_n h_n\{d_n - g_n(\mathbf{x})\}$



Condition at saddle point

$$\frac{\partial L(\mathbf{x}^*,\mathbf{h}^*)}{\partial x_m} = 0 \qquad \forall m$$

$$h_n \frac{\partial L(\mathbf{x}^*,\mathbf{h}^*)}{\partial h_n} = 0 \quad \text{and} \quad \frac{\partial L(\mathbf{x}^*,\mathbf{h}^*)}{\partial h_n} \le 0 \qquad h_n \ge 0 \;, \forall n$$

$$\frac{\partial Z(\mathbf{x}^*)}{\partial x_m} - \sum_n h_n \frac{\partial g_n(\mathbf{x}^*)}{\partial x_m} = 0 \qquad h_n \ge 0 \quad \text{and} \quad h_n\{d_n - g_n(x^*)\} = 0 \qquad \forall m, \forall n$$

$$g_n(x^*) \ge d_n$$

Solution of UE

From (1) $\begin{cases} \dfrac{\partial L}{\partial f_k^{rs}} = 0 & \left(f_k^{rs} \geq 0\right) \\ \dfrac{\partial L}{\partial f_k^{rs}} \geq 0 & \left(f_k^{rs} = 0\right) \end{cases}$ From (2) $\sum_{k \in K_{rs}} f_k^{rs} - Q_{rs} = 0$

$$\frac{\partial L}{\partial f_k^{rs}} = \frac{\partial}{\partial f_k^{rs}} \left\{ \sum_{a \in A} \int_0^{x_a} t_a(w)dw \right\} - \frac{\partial}{\partial f_k^{rs}} \left\{ \sum_{rs \in \Omega} \lambda_{rs} \left(\sum_{k \in K_{rs}} f_k^{rs} - Q_{rs} \right) \right\}$$

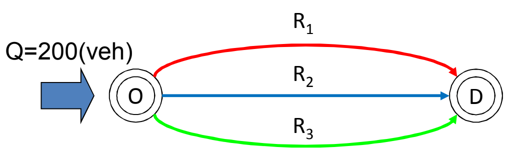
$$= \sum_{a \in A} \frac{d}{dx_a} \left(\int_0^{x_a} t_a(w)dw \right) \frac{\partial x_a}{\partial f_k^{rs}} - \lambda_{rs} = \sum_{a \in A} \delta_{a,k}^{rs} t_a(x_a) - \lambda_{rs} = c_k^{rs} - \lambda_{rs} \equiv c_k^{rs} - c_{rs}$$

where $\dfrac{\partial x_a}{\partial f_k^{rs}} = \dfrac{\partial}{\partial f_k^{rs}} \left(\sum_{rs \in \Omega} \sum_{k \in K_{rs}} \delta_{a,k}^{rs} f_k^{rs} \right) = \delta_{a,k}^{rs}$

$$\therefore \begin{cases} c_k^{rs} = c_{rs} & \left(f_k^{rs} > 0\right) \\ c_k^{rs} \geq c_{rs} & \left(f_k^{rs} = 0\right) \end{cases}$$

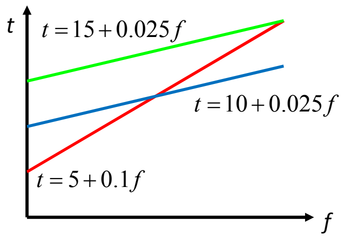
Example

Network & Demand



How much are link flows?

Link performance function



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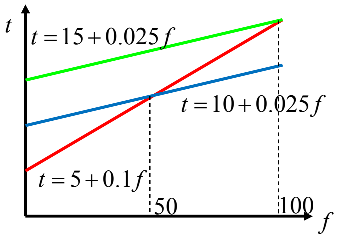
Solution



$$\min Z_p = \int_0^{f_1} t_1(w)dw + \int_0^{f_2} t_2(w)dw + \int_0^{f_3} t_3(w)dw$$
$$= 5f_1 + 0.05f_1^2 + 10f_2 + 0.0125f_2^2 + 15f_3 + 0.0125f_3^2$$
$$\text{s.t.} \quad \sum_{k=1}^{3} f_k = 200 \quad \text{and} \quad f_1 \geq 0 \,,\; f_2 \geq 0 \,,\; f_3 \geq 0$$

Lagrangian function

$$L(\mathbf{f}, \lambda) = 5f_1 + 0.05f_1^2 + 10f_2 + 0.0125f_2^2 + 15f_3 + 0.0125f_3^2 - \lambda\left(\sum_{k=1}^{3} f_k - 200\right)$$

Kuhn-Tucker condition

$$\begin{cases} 5 + 0.1f_1 = \lambda & (f_1 > 0) \\ 5 + 0.1f_1 \geq \lambda & (f_1 = 0) \end{cases} \quad \begin{cases} 10 + 0.025f_2 = \lambda & (f_2 > 0) \\ 10 + 0.025f_2 \geq \lambda & (f_2 = 0) \end{cases} \quad \begin{cases} 15 + 0.025f_3 = \lambda & (f_3 > 0) \\ 15 + 0.025f_3 \geq \lambda & (f_3 = 0) \end{cases}$$

$$\sum_{k=1}^{3} f_k = 200$$

Solution (by searching)

$$f_1 = 80 \,,\; f_2 = 120 \,,\; f_3 = 0 \,,\; \lambda = 13$$

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Formulation of SO(システム最適)

- At system optimum (**SO**), the total travel time (or generalized cost) of all vehicles is smallest

$$\min Z_s = \sum_{a \in A} \int_0^{x_a} w t_a(w) dw$$
$$s.t. \quad \sum_{k \in K_{rs}} f_k^{rs} - Q_{rs} = 0 \quad , {}^{\forall} k \in K_{rs} \,, {}^{\forall} rs \in \Omega$$
$$f_k^{rs} \geq 0$$
$$c_k^{rs} = \sum_{a \in A} \delta_{a,k}^{rs} t_a(x_a)$$
$$x_a = \sum_{k \in K_{rs}, rs \in \Omega} \delta_{a,k}^{rs} f_k^{rs}$$

Lagrangian function

$$L(\mathbf{f},\boldsymbol{\lambda}) = Z_s(\mathbf{f}) - \sum_{rs\in\Omega} \lambda_{rs} \left(\sum_{k\in K_{rs}} f_k^{rs} - Q_{rs} \right)$$

Kuhn-Tucker (optimal) condition

$$\begin{cases} (1)\ f_k^{rs} \dfrac{\partial L(\mathbf{f}^*,\boldsymbol{\lambda}^*)}{\partial f_k^{rs}} = 0 \ \text{ and } \ \dfrac{\partial L(\mathbf{f}^*,\boldsymbol{\lambda}^*)}{\partial f_k^{rs}} \geq 0 \ ,^{\forall} k \in K_{rs} \ ,^{\forall} rs \in \Omega \\ (2)\ \dfrac{\partial L(\mathbf{f}^*,\boldsymbol{\lambda}^*)}{\partial \lambda_{rs}} = 0 \ ,^{\forall} rs \in \Omega \end{cases}$$

Solution of SO

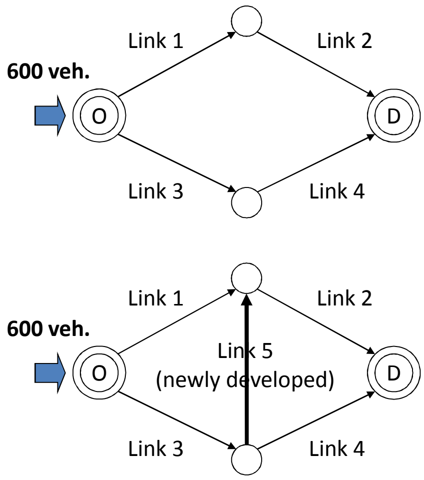
$$\text{From (1)} \begin{cases} \dfrac{\partial L}{\partial f_k^{rs}} = 0 \ \left(f_k^{rs} \geq 0\right) \\ \dfrac{\partial L}{\partial f_k^{rs}} \geq 0 \ \left(f_k^{rs} = 0\right) \end{cases} \qquad \text{From (2)} \ \sum_{k\in K_{rs}} f_k^{rs} - Q_{rs} = 0$$

$$\frac{\partial L}{\partial f_k^{rs}} = \sum_{a\in A} \delta_{a,k}^{rs} \left\{ t_a(x_a) + \left. \frac{dt_a(x)}{dw} \right|_{w=x_a} \right\} - \lambda_{rs}$$

$$\therefore c_k^{rs} = \lambda_{rs} + \boxed{\left. \frac{dt_a(x)}{dw} \right|_{w=x_a}}$$

Extra travel time by one additional user

Interesting phenomenon: Braess's Paradox

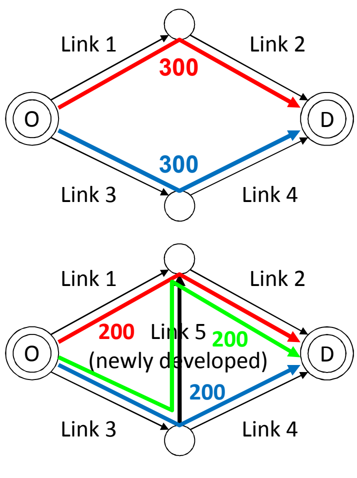


Link performance function

$$\begin{cases} t_1 = 50 + 0.01x_1 \\ t_2 = 0.1x_2 \\ t_3 = 0.1x_3 \\ t_4 = 50 + 0.01x_4 \\ t_5 = 10 + 0.01x_5 \end{cases}$$

Derive link flow parameters at UE & SO state

UE state



Travel time = 83

Total travel time = 83 × 600 = 49,800

Travel time = 92

Total travel time = 92 × 600 = 55,200

Bad effect of road construction!

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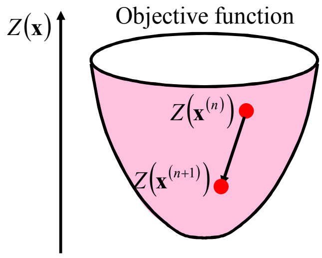
General algorithm for solution of NLOP by computer



Method of steepest descent (最急降下法)

(1) Find a direction to make the value of OJB function much smaller
(2) Find a optimal step size in descending direction

$$\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \alpha\left\{\mathbf{y} - \mathbf{x}^{(n)}\right\}$$

$$\mathbf{y} = \nabla Z\left(\mathbf{x}^{(n)}\right)$$

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Frank-Wolfe Method: How to find descending direction?

$$\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \alpha\left\{\mathbf{y} - \mathbf{x}^{(n)}\right\}$$

$$Z(\mathbf{y}) \cong Z'(\mathbf{y}) = Z\left(\mathbf{x}^{(n)}\right) + \nabla Z\left(\mathbf{x}^{(n)}\right)^T\left(\mathbf{y} - \mathbf{x}^{(n)}\right) \quad \text{Linear approximation}$$

$$= Z\left(\mathbf{x}^{(n)}\right) + \sum_{a\in A}\left(y_a - x_a^{(n)}\right)\frac{\partial Z\left(\mathbf{x}^{(n)}\right)}{\partial x_a^{(n)}} = Z\left(\mathbf{x}^{(n)}\right) + \sum_{a\in A}\left(y_a - x_a^{(n)}\right)t_a\left(x_a^{(n)}\right)$$

$$= \underbrace{Z\left(\mathbf{x}^{(n)}\right) - \sum_{a\in A} x_a^{(n)} t_a\left(x_a^{(n)}\right)}_{\text{Constant}} + \sum_{a\in A} y_a t_a\left(x_a^{(n)}\right)$$

$$\Rightarrow \min Z'(\mathbf{y}) = \sum_{a\in A} y_a t_a\left(x_a^{(n)}\right) \quad \text{s.t.} \sum_{k\in K_{rs}} f_k^{rs} - Q_{rs} = 0 \text{ and } y_a = \sum_{k\in K_{rs}}\sum_{rs\in\Omega}\delta_{a,k}^{rs} f_k^{rs}$$

Calculate $\mathbf{y}$ minimizing total travel time in condition that $x_a = x_a^{(n)}$

All-or-nothing assignment of all OD volume to the shortest path when $x_a = x_a^{(n)}$

Minimized $\mathbf{y}$ gives the steepest descent of $\nabla Z\left(\mathbf{x}^{(n)}\right)$

#Easy computing

Frank-Wolfe Method: Algorithm

Step0 Set initial link traffic flow $x_a^{(1)}$ by all-or-nothing assignment in condition of $t_a(0)$, and set iteration number n=1

Step1 Calculate $t_a^{(n)}$ by link performance function

Step2 Calculate $y_a^{(n)}$, which is link traffic flow based on all-or-nothing assignment to the shortest path in condition of $t_a^{(n)}$, and calculate descending direction vector $d_a^{(n)} = y_a^{(n)} - x_a^{(n)}$

Step3 Find step size $\zeta^{(n)}$ to minimize $Z^{(n+1)}$ by non-linear line searching

$$x_a^{(n+1)} = x_a^{(n)} + \zeta^{(n)} d_a^{(n)}$$

$$Z^{(n+1)} = \sum_{a\in A} \int_0^{x_a^{(n+1)}} t_a(w)dw$$

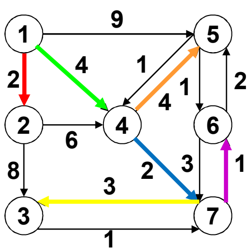
Step4 Calculate $x_a^{(n+1)}$ using $\zeta^{(n)}$

Step5 If $x_a^{(n+1)}$ is close to $x_a^{(n)}$, $x_a^{(n+1)}$ is the solution

If not, back to Step1 $n \rightarrow n+1$

Shortest path searching (最短経路探索) algorithm

Dijkstra method



step	link		label list							Predecessor list							min C
			node							node							
	O	D	1	2	3	4	5	6	7	1	2	3	4	5	6	7	
1			*0*	∞	∞	∞	∞	∞	∞	0	0	0	0	0	0	0	1
2	1	2,4,5	*0*	2	∞	4	9	∞	∞	0	1	0	1	1	0	0	2
3	2	3,4	*0*	*2*	10	4	9	∞	∞	0	1	2	1	1	0	0	4
4	4	5,7	*0*	*2*	10	*4*	8	∞	6	0	1	2	1	4	0	4	7
5	7	3,6	*0*	*2*	9	*4*	8	7	*6*	0	1	7	1	4	7	4	6
6	6	5,7	*0*	*2*	9	*4*	8	*7*	*6*	0	1	7	1	4	7	4	5
7	5	4,6	*0*	*2*	9	*4*	*8*	*7*	*6*	0	1	7	1	4	7	4	3

Stochastic user equilibrium (SUE) (確率的利用者均衡)

- Limitations of the assumption of UE
  - Any user do not have a full information (**完全情報**) of all link costs in the network
- Introduction of uncertainty (**不確実性**) of link costs
  - The link cost perceived by user varies stochastically
  - User chooses a path with minimum utility (cost)

- At stochastic user equilibrium (**SUE**), all travelers believe that they cannot shorten their OD travel time (or generalized cost) by transferring to any different path

Formulation of SUE(1)

Path utility: $U_k^{rs} = -c_k^{rs} + \underbrace{\xi_k^{rs}}_{\text{Error term}}$, $c_k^{rs} = \sum_{a\in A} t_a \delta_{a,k}^{rs}$

Choice probability of path k of OD rs (Random utility theory)

$$P_k^{rs} = \Pr\left[U_k^{rs} \ge \max_{k'\neq k}\left(U_{k'}^{rs}\right)\right] = \Pr\left[c_k^{rs} - \xi_k^{rs} \ge \max_{k'\neq k}\left(c_{k'}^{rs} - \xi_{k'}^{rs}\right)\right]$$

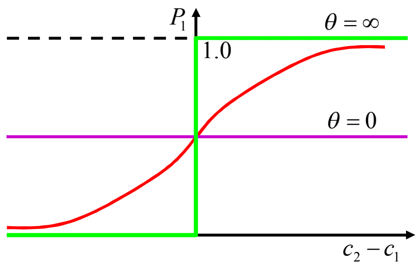
Choice probability if error term is determined by Weibul distribution $W(0,\theta)$

$$P_k^{rs} = \frac{\exp\left(-\theta c_k^{rs}\right)}{\sum_{k'\in K_{rs}} \exp\left(-\theta c_{k'}^{rs}\right)}$$

Expected path & link flow

$$E\left(f_k^{rs}\right) = Q_{rs} P_k^{rs}$$

$$E\left(x_a\right) = \sum_{rs\in\Omega} \sum_{k\in K_{rs}} E\left(f_k^{rs}\right) \delta_{a,k}^{rs}$$



Formulation of SUE(2)

Optimization problem

$$\min Z(\mathbf{f}) = \sum_{a\in A}\int_0^{x_a} t_a(w)dw - \frac{1}{\theta}\sum_{rs\in\Omega} Q_{rs} H_{rs}(\mathbf{f}^{rs})$$

Entropy term: $H_{rs}(\mathbf{f}^{rs}) \equiv -\sum_{k\in K_{rs}} P_k^{rs} \ln P_k^{rs} = -\sum_{k\in K_{rs}} \frac{f_k^{rs}}{Q_{rs}} \ln \frac{f_k^{rs}}{Q_{rs}}$

s.t. $\sum_{k\in K_{rs}} f_k^{rs} - Q_{rs} = 0$, $x_a = \sum_{k\in K_{rs}, rs\in\Omega} \delta_{a,k}^{rs} f_k^{rs}$, $f_k^{rs} \geq 0$

Solution of SUE

Lagrangian function

$$L(\mathbf{f},\boldsymbol{\lambda}) = Z(\mathbf{f}) - \sum_{rs\in\Omega}\lambda_{rs}\left(\sum_{k\in K_{rs}} f_k^{rs} - Q_{rs}\right),\ f_k^{rs} \geq 0$$

Kuhn-Tucker condition

$$f_k^{rs}\frac{\partial L(\mathbf{f}^*,\boldsymbol{\lambda}^*)}{\partial f_k^{rs}} = 0 \text{ and } \frac{\partial L(\mathbf{f}^*,\boldsymbol{\lambda}^*)}{\partial f_k^{rs}} \geq 0$$

$$\frac{\partial L}{\partial f_k^{rs}} = c_k^{rs} - \lambda_{rs} - \frac{1}{\theta}\left(\ln f_k^{rs} + 1 - \ln Q_{rs}\right) = 0 \Rightarrow f_k^{rs} = Q_{rs}\exp\left(-\theta c_k^{rs}\right)\exp(\theta\lambda_{rs} - 1)$$

$$Q_{rs} = \sum_{k\in K_{rs}} f_k^{rs} = Q_{rs}\exp(\theta\lambda_{rs} - 1)\sum_{k\in K_{rs}}\exp\left(-\theta c_k^{rs}\right)$$

$$\therefore \exp(\theta\lambda_{rs} - 1) = \frac{1}{\sum_{k\in K_{rs}}\exp\left(-\theta c_k^{rs}\right)}$$

$$\therefore f_k^{rs} = Q_{rs}\frac{\exp\left(-\theta c_k^{rs}\right)}{\sum_{k\in K_{rs}}\exp\left(-\theta c_k^{rs}\right)}$$

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Meaning of entropy term

$$H_{rs}(\mathbf{f}^{rs}) \equiv -\sum_{k \in K_{rs}} P_k^{rs} \ln P_k^{rs} = -\sum_{k \in K_{rs}} \frac{f_k^{rs}}{Q_{rs}} \ln \frac{f_k^{rs}}{Q_{rs}}$$

$$\max H_{rs}(\mathbf{f}^{rs}) \rightarrow f_1^{rs} = \cdots = f_k^{rs} = \cdots = f_K^{rs} = \frac{Q_{rs}}{K}$$

$$\min H_{rs}(\mathbf{f}^{rs}) \rightarrow f_k^{rs} = Q_{rs}, f_1^{rs} = \cdots = f_K^{rs} = 0$$

Entropy maximization = Increasing evenness

SUE = minimizing travel time considering unevenness

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Consideration of dynamic aspect

- Dynamic problem: OD volume and link flow are expressed by a function of time t
- Dynamic System Optimum (**DSO**)
  - Total travel time (or generalized cost) of all vehicles in focused time duration is smallest
- Dynamic User Optimum (**DUO**)
  - All travelers choose the path with smallest travel time (or generalized cost) instantaneously at focused time
- Dynamic User Equilibrium (**DUE**)
  - All travelers choose the path with smallest travel time (or generalized cost)

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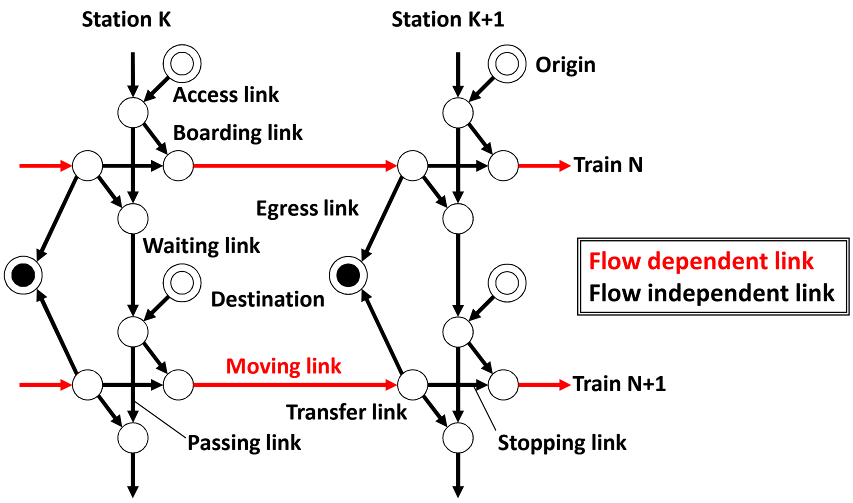
Application to tempo-spatial network – train schedule problem



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Formulation (UE)

$$\min Z_P = \sum_{a\in A} \int_0^{x_a} t_a(w)dw$$

$$s.t.\ \sum_{k\in K_{rs}} f_k^{rs} - Q_{rs} = 0,\ ^{\forall}k \in K_{rs},\ ^{\forall}rs \in \Omega$$

$$f_k^{rs} \geq 0$$

$$c_k^{rs} = \sum_{a\in A} \delta_{a,k}^{rs} t_a(x_a)$$

$$x_a = \sum_{k\in K_{rs}, rs\in\Omega} \delta_{a,k}^{rs} f_k^{rs}$$

Link performance function

Moving link: $t_a(w) = T_a\left(1 + \lambda \frac{w}{\mu C - w}\right)$ = Congestion disutility

Waiting link: $t_a(w) = \alpha W_a$

Transfer link: $t_a(w) = \beta F_a$

Travel time (T_a), Train capacity (C), Waiting time (W_a), Transfer time (F_a)